Pergamon

Int. J. Solids Structures Vol. 34, No. 18, pp. 2239–2254, 1997

0020–7683/97 $17.00 + .00

PII: S0020–7683(96)00162–X

THERMODYNAMIC FORMULATION OF CONSTITUTIVE EQUATIONS AND APPLICATION TO THE VISCOPLASTICITY AND VISCOELASTICITY OF METALS AND POLYMERS

JEAN-LOUIS CHABOCHE
O.N.E.R.A., 29, avenue de la Division Leclerc, 92320 Châtillon, France

(*Received* 10 *April* 1996; *in revised form* 29 *July* 1996)

Abstract—Elastoviscoplasticity for metallic materials and viscoelasticity for polymers are generally treated as completely independent processes. In the present paper we intend to discuss some possible bridges between the two kind of constitutive theories. Two aspects are more specifically addressed: (i) the classical thermodynamics of irreversible processes, using material state variables, is further extended in order to incorporate more easily dynamic and static recovery effects in the kinematic hardening evolutionary equations; (ii) the qualitative and quantitative equivalence between elastoviscoplasticity and viscoelasticity is discussed both in terms of the simplest linear model and for a more complicated non-linear situation. In fact, it is demonstrated that superposing several non-linear kinematic hardening and static recovery rules allows to model a viscoelastic behavior. One of the advantages of the approach is to describe easily the cyclic loading situations, as shown by comparison with experiments made on a glass–epoxy composite. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Many constitutive equations have been developed during the past fifteen years for the description of cyclic viscoplasticity of metallic materials. Most often they use hardening state variables related to the present dislocation microstructures, incorporating the description of complex phenomena such as Bauschinger effects, creep and relaxation, thermal recovery effects, cyclic hardening or softening, various memorization effects, etc.

On the other hand, polymer like materials are generally considered to obey viscoelasticity rules. The classical approaches use integral equations based on an hereditary formulation of irreversible thermodynamics. In some cases other theories have been developed, using spectral decompositions of the strain but still in the framework of viscoelasticity.

The twofold purpose of the present article is to: (i) reexamine the classical framework of thermodynamics based on internal variables, introducing a new way of generalization that offers additional flexibilities; (ii) discuss the possible links that can be constructed between elastoviscoplasticity and viscoelasticity; apply to polymer based materials the constitutive equations initially developed for metals.

The formulations considered here are deliberately macroscopic and come within the framework of continuum mechanics, which is the basic tool of all modern structural computation methods. So while the real material may have physical discontinuities at the various microstructural scales, these scales are not expressed explicitly but are described globally at the level of a homogenized bulk element of the material (representative volume element). This very macroscopic level of description is one of the reasons that inelastic properties of real parts can be analyzed while still hoping to tie them in with polymers.

In the following, we will limit ourselves to the presentation of the constitutive equations without damage and within the assumption of small deformations, while first emphasizing internal variables generally used for metals (Section 2). The next section then presents a new general formalism constructed in the framework of Thermodynamics of Irreversible Processes. In particular, a new way of generalizing the formal framework of Generalized Standard Materials, based on the idea of multiple potentials, is explained and used for the high-temperature cyclic viscoplasticity laws. The case of polymer materials with the different treatment they require is addressed in Section 4. We then show the usable links between the theories originally constructed for metals and models of use for polymers. In particular a set of viscoplastic constitutive equations previously developed for metallic materials is applied with some success to experiments made on a glass–epoxy composite.

2. MAIN OBSERVATIONS FOR METALLIC MATERIALS AND ASSOCIATED INTERNAL VARIABLES

When a metal is strained beyond its elastic range, irreversible deformations appear that remain after the load is removed. These deformations are due to slippage of the atomic planes, i.e., to dislocation movements and to the appearance of new dislocations. These processes, which are very complex at the microstructural scale in polycrystalline and polyphased materials, are reflected by macroscopic plastic strains. In small deformations, it is assumed that the total strain can be separated into elastic and plastic (or viscoplastic) strains: $\varepsilon = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_p$.

The increase in the dislocation density associates a hardening effect with the monotonic plastic strain. The dislocations interlock, lose their mobility by piling up on obstacles or forming cells, and the stress needed to initiate their movement again increases. Under cyclic loading, the hardening effects become even more complex.

At ordinary temperatures, it can generally be considered that the dislocation motions are independent of time and rate. The material is then considered in the "rate-independent plasticity" framework. At high temperatures, on the other hand, viscosity effects appear due to the thermal agitation processes that promote the dislocation deviated slip and climb mechanisms. In the case of creep, the hardening is observed as a decrease in the strain rate in the course of primary creep.

Static and thermal recovery processes occur at high temperatures. They are characterized by a slow, partial restoration of the plastic strain, and by a progressive recovery of the hardening, i.e., a return to the original microstructure (thermal agitation promotes dislocation annihilation).

In order to describe the above observed facts we generally use the notion of an elasticity domain in the stress space and various internal state variables that are briefly introduced now as internal stresses.

The idea of internal stress is already an old one. It provides a better way of expressing the observed macroscopic properties, in particular in polycrystalline and polyphase materials in which the exponent of the secondary creep law is very high. Introducing an internal stress σ_i reduces this exponent considerably. It can be measured by tests including successive unloadings:

$$\dot{\varepsilon}_p = \left(\frac{\sigma - \sigma_i}{K}\right)^n. \tag{1}$$

The internal stress can be broken down into a sum of elementary stresses. In the multiaxial case, the strain rate norm $\dot{p}$ can be written in terms of the viscous stress σ_v (Lemaître and Chaboche, 1985; Chaboche, 1983):

$$\sigma_v = J(\boldsymbol{\sigma} - \mathbf{X}) - k - R - R^* \qquad \dot{p} = \left\langle \frac{\sigma_v}{K} \right\rangle^n \tag{2}$$

in which J is a distance in the stress space. For a material meeting the Von Mises criterion, we use $J(\boldsymbol{\sigma} - \mathbf{X}) = [(3/2)(\boldsymbol{\sigma}' - \mathbf{X}'):(\boldsymbol{\sigma}' - \mathbf{X}')]^{1/2}$, in which $\boldsymbol{\sigma}'$ and $\mathbf{X}'$ are deviators of the tensors $\boldsymbol{\sigma}$ and $\mathbf{X}$. The elementary internal stresses are:

- **X** is a rank-two tensor called the kinematic stress, or back stress. It corresponds to the short, medium and long-range interactions: intergranular stresses induced by inhomogeneities in the plastic strains from one grain to another, interaction between dislocations and precipitates, and so forth. Several analogous variables can be superimposed. Their variation equations are either linear (Prager's law) or non-linear. One of the most widely used models contains a linear hardening term and a recall, or dynamic recovery, term. This will be developed in Section 3.4.
- k is the Orowan's isotropic (or scalar) stress corresponding to the initial yield strength of the material, and depends among other things on the volume fraction of precipitates, on their size, and on the initial density of the dislocations. Of course it also depends on the temperature.
- R is the variation of the Orowan's stress induced by the plastic strain. This is directly related to the increase in dislocation density, but may depend on the configurations assumed by the dislocations, with the creation of dislocation cells, and the size and fineness of these cells. More or less complex models have been developed to reflect cyclic hardening or softening, cyclic hardening memory effects, and so on.
- R^* is used to describe the drag mechanisms due to the interstitial atoms, or resistance variations due to changes in the material's microstructure and various metallurgical phenomena like dissolution of precipitates, reprecipitation and coalescence, aging effects, often induced by temperature changes.

The evolution laws of these variables conventionally used for metals in the framework of plasticity and viscoplasticity will be developed in their general form in Section 3.4. In Chaboche (1989), Nouailhas (1989), and Ohno (1990), McDowell (1992), the reader can find discussions on the most recent developments, in particular for cyclic and/or anisothermal loadings. Let us note that some viscoplasticity theories (Miller, 1976; Bodner, 1975; Walker, 1981) do not use an elasticity domain. In these theories the effects associated to R or R^* are introduced in the drag stress K in eqn (1).

3. GENERAL FORMALISM

3.1. *Yield surface and plastic flow*

The classical theories of plasticity assume there exists a yield surface defined in the stress space, which we will denote:

$$f = f(\boldsymbol{\sigma}; V_k) \leqslant 0 \tag{3}$$

in which V_k represents the state (hardening) variables giving the shape and position of the $f = 0$ surface at each instant in time. The stress states inside the yield surface, where $f < 0$ causes only reversible (elastic) strains. The area where $f \leqslant 0$ may be reduced to a point, if necessary.

When $f = 0$, the plastic flow may occur, as long as the loading (i.e., the stress change) is directed outward from the domain. The normality rule stemming from Hill's maximum work principle is expressed:

$$\mathrm{d}\boldsymbol{\varepsilon}_p = \mathrm{d}\lambda \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{4}$$

in which $\partial f/\partial \boldsymbol{\sigma}$ is the normal to $f = 0$, and $\mathrm{d}\lambda$ the plastic multiplier defined below.

In the case of rate-independent plasticity, the stress cannot leave the surface and we always have $f = 0$ as long as there is no unloading. Knowing the hardening laws, i.e., the laws of the V_k variables, the rate-independent plasticity *consistency condition*, i.e., $f = \mathrm{d}f = 0$, can be used to determine the plastic multiplier $\mathrm{d}\lambda$.

On the other hand, in the case of viscoplasticity, the stress state may range beyond the yield surface ($f > 0$), and the strain rate increases with the distance from the surface, the viscous stress σ_v. In this case, the multiplier is determined by a given function $\mathrm{d}\lambda/\mathrm{d}t$ of the viscous stress, as for example in the case of eqn (2) above.

3.2. *Classical thermodynamic formalism*

Though it is not entirely indispensable for purely mechanical purposes, the laws of plasticity/viscoplasticity can be expressed in a general thermodynamic framework (Germain, 1973) within the hypothesis of the local state. The state variables are:

- observable variables, i.e., the temperature and strain. The more direct expression we follow here is based on the elastic strain as observable variable;
- internal or hidden variables, denoted V_k. These variables are the lump sum of the entire previous history of change of the material.

The thermodynamic formalism is based on the assumption of the existence of two potentials (Lemaître and Chaboche, 1985):

- *State potential*, or thermodynamic potential, which is sufficient to describe all the reversible processes (here we will use the Helmholtz free energy). This depends on all the state variables: $\psi = \psi(\boldsymbol{\varepsilon}_e, T, V_k)$. The First and Second Principles of Thermodynamics are used to derive the laws of thermoelasticity:

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}_e} \qquad S = \frac{\partial \psi}{\partial T} \qquad A_k = \frac{\partial \psi}{\partial V_k} \tag{5}$$

in which $\boldsymbol{\sigma}$ is the stress tensor (the first equation above is thus the general form of Hooke's law), and S the entropy. $\boldsymbol{\sigma}$ and S are thus considered as dual variables, thermodynamic forces associated respectively with $\boldsymbol{\varepsilon}_e$ and T. By analogy, we define $A_k = \partial\psi/\partial V_k$ to be the forces associated with V_k.

- *Dissipation potential* to describe the irreversible phenomena that cause a given type of dissipation. We assume that this potential is expressed in the space of thermodynamic forces, $\varphi = \varphi(\boldsymbol{\sigma}, A_k; T, V_k)$, and that the state variables can act as parameters. The normal dissipative laws (or generalized normality laws) are then expressed:

$$\mathrm{d}\boldsymbol{\varepsilon}_p = \mathrm{d}\lambda \frac{\partial \varphi}{\partial \boldsymbol{\sigma}} \qquad \mathrm{d}V_k = -\mathrm{d}\lambda \frac{\partial \varphi}{\partial A_k}. \tag{6}$$

This generalization of the normality rule defines the formal framework of "generalized standard materials" (Halphen and Nguyen, 1975; Germain *et al.*, 1983). Let us immediately point out that this generalization can theoretically be used to verify the inequality of the Second Principle automatically, which is written here (q is the heat flux):

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p - A_k \dot{V}_k - \frac{1}{T} q \cdot \operatorname{grad} T \geqslant 0. \tag{7}$$

3.3. *Extension of the formalism with multiple potentials*

Staying within the strict framework of generalized standard materials imposes limitations that are sometimes too restrictive, due to the use of a single global potential. Hereafter, we present a somewhat broader framework in which we accept the possibility of several potentials defined independently, corresponding to different processes (Chaboche *et al.*, 1995). This is because we have difficulty understanding why phenomena as different as thermal dissipation, plastic dissipation, damage, or metallurgical variations should all derive from a single potential. This kind of approach is already implied in a certain number of recent works (Benallal, 1989; Hansen and Schreyer, 1992) and stems also from the classical formalism of plasticity with multiple criteria (Mandel, 1971).

Table 1. The three potentials and the specific forces and parameters

Potential	Multiplier	⟨⟨Force⟩⟩	Specific ⟨⟨force⟩⟩				Parameters			
Ω_p	$\dot\lambda_p$	A_p^k	σ	X_k	R_k		α_k	r_k	d_j	T
Ω_s	$\dot\lambda_s$	A_p^k		X_k	R_k		α_k	r_k	d_j	T
Ω_d	$\dot\lambda_d$	A_d^j				Y_j			d_j	T
	Corresponding rates		$\dot{\boldsymbol{\varepsilon}}_p$	$\underbrace{-\dot\alpha_k \quad -\dot r_k}_{-\dot V_p^k}$		$\dot V_d^j$		d_j		

Leaving aside the thermal dissipation potential, which leads to the Fourier's law, we therefore define (Table 1) three independent potentials expressed in the thermodynamic forces space:

- Ω_p, associated with the plastic strain and hardening process that accompanies it;
- Ω_s, the slow (static) microstructural evolution potential, such as the static recovery and aging (dissolution of precipitates or precipitation, for example);
- Ω_d, the damage potential, while the damage mechanisms are considered as acting on a scale other than those of the plastic strains.

Table 1 also indicates the thermodynamic forces (or affinities) the potentials depend on, in addition to the stress:

- A_p^k, the hardening forces associated with the variables V_p^k. In practice, the specific forces will be X_k and R_k, the kinematic and isotropic hardening variables, expressed in the stress space for defining the center and size of the real yield surface.
- A_d^j, the damage forces, or the energy released by the increase of the variable V_d^j.

The main hypothesis of this extension is to express the generalized normality rule in the following form:

$$\dot{\boldsymbol{\varepsilon}}_p = \dot\lambda_p \frac{\partial \Omega_p}{\partial \boldsymbol{\sigma}} + \dot\lambda_s \frac{\partial \Omega_s}{\partial \boldsymbol{\sigma}} + \dot\lambda_d \frac{\partial \Omega_d}{\partial \boldsymbol{\sigma}}$$

$$-\dot V_k^p = \dot\lambda_p \frac{\partial \Omega_p}{\partial A_p^k} + \dot\lambda_s \frac{\partial \Omega_s}{\partial A_p^k} + \dot\lambda_d \frac{\partial \Omega_d}{\partial A_p^k}$$

$$-\dot V_d^j = \dot\lambda_p \frac{\partial \Omega_p}{\partial A_d^j} + \dot\lambda_s \frac{\partial \Omega_s}{\partial A_d^j} + \dot\lambda_d \frac{\partial \Omega_d}{\partial A_d^j}. \tag{8}$$

Of course, with the dependencies in Table 1, certain derivatives are zero, and we get:

$$\dot{\boldsymbol{\varepsilon}}_p = \dot\lambda_p \frac{\partial \Omega_p}{\partial \boldsymbol{\sigma}}$$

$$-\dot V_k^p = \dot\lambda_p \frac{\partial \Omega_p}{\partial A_p^k} + \dot\lambda_s \frac{\partial \Omega_s}{\partial A_p^k}$$

$$-\dot V_d^j = \dot\lambda_d \frac{\partial \Omega_d}{\partial A_d^j}. \tag{9}$$

The multipliers $\dot\lambda_p$, $\dot\lambda_s$, $\dot\lambda_d$ can be defined independently as Lagrange multipliers (Hansen and Schreyer, 1992; Chow and Wei, 1991). In fact, for viscoplasticity, we can take for $\dot\lambda_p$ a scalar function of the viscous stress given by the distance f from the stress state to the elastic domain, in the form of a power function, for example. The simplest definition for $\dot\lambda_s$ is to choose a constant value of unity (Chaboche *et al.*, 1995). The choice of $\dot\lambda_d$ is not discussed here. The appendix gives a few elements to support this broadened formulation, in comparison with the strict framework of the generalized standard materials.

3.4. *Particularized choices for isotropic and kinematic hardening*

All of the hardening laws ordinarily used for metals, mentioned in Section 2 above, can easily be expressed in the thermodynamic framework introduced here. More precisely, we use tensor variables $\boldsymbol{\alpha}_i$, homogeneous in strain, to define the kinematic hardening, and one (or more) scalar variables r to express the isotropic hardening. The free energy is assumed to be of the form:

$$\psi = \frac{1}{2}\boldsymbol{\varepsilon}_e : \Lambda : \boldsymbol{\varepsilon}_e + \sum_i \boldsymbol{\alpha}_i : \mathbf{C}_i : \boldsymbol{\alpha}_i + g(r) \tag{10}$$

in which Λ and $\mathbf{C}_i$ are fourth rank tensors, the former representing the elastic stiffness tensor. The state laws are:

$$\boldsymbol{\sigma} = \Lambda : \boldsymbol{\varepsilon}_e \qquad \mathbf{X}_i = \mathbf{C}_i : \boldsymbol{\alpha}_i \qquad R = \frac{\partial g}{\partial r} \tag{11}$$

in which we have assumed that the thermodynamic forces associated with $\boldsymbol{\alpha}_i$ and r are $\mathbf{X}_i$ and R, respectively. Moreover, the actual center of the yield surface is considered to be the sum:

$$\mathbf{X} = \sum_i \mathbf{X}_i \tag{12}$$

and R gives the variation in its size (since the initial state). The potentials Ω_p and Ω_s can be taken in the form:

$$\Omega_p = f + \frac{1}{2}\sum_i \| X_i \|^2_{Q_i} + G(R) \tag{13}$$

$$\Omega_s = \sum_i \frac{a_i}{m_i+1}\left(\frac{\|X_i\|_{\Gamma_i}}{a_i}\right)^{m_i+1} + \frac{b}{m+1}\left(\frac{R}{b}\right)^{m+1} \tag{14}$$

in which the norm $\|X\|_\Gamma$ is defined by $\|X\|_\Gamma = (\mathbf{X} : \boldsymbol{\Gamma} : \mathbf{X})^{1/2}$, in which $\boldsymbol{\Gamma}$ is a fourth rank tensor. The first expression uses the yield surface range, meeting Hill's criterion (by generalizing (2)), still with the same notation for the corresponding norms (in some transformed stress space):

$$f = \|\sigma - X\|_M - R - k. \tag{15}$$

Moreover, the viscoplastic multiplier λ_p is expressed in the form of a power function:

$$\dot{\lambda}_p = \left\langle \frac{f}{K} \right\rangle^n \tag{16}$$

where $f < 0$ corresponds to the elastic domain $\dot{\lambda}_p = 0$, and $f > 0$ gives the distance between the current stress state and the boundary $f = 0$. We further assume that the static multiplier is equal to unity.

Applying the normality rule (6) provides the following equations:

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}_p \frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda}_p \frac{\mathbf{M}:(\boldsymbol{\sigma}-\mathbf{X})}{\|\sigma - X\|_M} \tag{17}$$

$$\dot{\boldsymbol{\alpha}}_i = \dot{\boldsymbol{\varepsilon}}_p - \dot{\lambda}_p \mathbf{Q}_i \mathbf{X}_i - \left(\frac{\|X_i\|_{\Gamma_i}}{a_i}\right)^{m_i} \frac{\boldsymbol{\Gamma}_i : \mathbf{X}_i}{\|X_i\|_{\Gamma_i}} \tag{18}$$

$$\dot{r} = \dot{\lambda}_p \left(1 - \frac{\partial G}{\partial R}\right) - \left(\frac{R}{b}\right)^m. \tag{19}$$

It can be seen that $\dot{\lambda}_p$ is nothing other than the norm of the plastic strain rate, defined by:

$$\dot{\lambda}_p = \dot{p} = (\dot{\boldsymbol{\varepsilon}}_p : \mathbf{M}^{-1} : \dot{\boldsymbol{\varepsilon}}_p)^{1/2} \tag{20}$$

and that the kinematic and isotropic hardening laws bring in two evanescent memory terms, one proportional to $\dot{p}$, or dynamic recovery (thanks to the additional terms in the potential Ω_p), and the other directly related to time, or static recovery (with potential Ω_s). The aging effects or other metallurgical evolution phenomena have not been considered here.

Using the relation between $\mathbf{X}_i$ and $\boldsymbol{\alpha}_i$, and choosing the scalar functions $g(r)$ and $G(R)$ (several choices are possible, mentioned in Chaboche, 1991), we easily obtain the hardening laws valid in isothermal mode:

$$\dot{\mathbf{X}}_i = \mathbf{C}_i : \dot{\boldsymbol{\varepsilon}}_p - \mathbf{D}_i : \mathbf{X}_i \dot{p} - \left(\frac{\|X_i\|_{\Gamma_i}}{a_i}\right)^{m_i} \frac{\bar{\boldsymbol{\Gamma}}_i : \mathbf{X}_i}{\|X_i\|_{\Gamma_i}} \tag{21}$$

$$\dot{R} = c\dot{p} - \mathrm{d}R\dot{p} - c\left(\frac{R}{b}\right)^m \tag{22}$$

in which we have let $\mathbf{D}_i = \mathbf{C}_i : \mathbf{Q}_i$ and $\bar{\boldsymbol{\Gamma}}_i = \mathbf{C}_i : \boldsymbol{\Gamma}_i$.

These laws were successfully used by Nouailhas (1990) for single-crystal alloys for turbine blades, in which the fourth rank tensors $\mathbf{M}$, $\mathbf{C}_i$, $\mathbf{D}_i$ and $\bar{\boldsymbol{\Gamma}}_i$ were defined by reference to the initial anisotropic state of the material (cubic symmetry). In the special case where these tensors are all proportional to the fourth rank unit tensor, we get the back-stress evolution laws of the isotropic material, i.e.

$$\dot{\mathbf{X}}_i = C_i \dot{\boldsymbol{\varepsilon}}_p - D_i \mathbf{X}_i \dot{p} - \left(\frac{\|X_i\|}{a_i}\right)^{m_i} \frac{\mathbf{X}_i}{\|X_i\|} \tag{23}$$

which are used in many applications on polycrystalline alloys (in that particular case C_i, D_i are scalars).

4. LINEAR AND NON-LINEAR VISCOELASTICITY

In this section, we will address a type of property that is theoretically very different from those mentioned above for metals. In theory, a viscoelastic material exhibits "delayed" elastic phenomena. In other words, the material's response is almost reversible, but only asymptotically, i.e., if one waits a long enough time. The closest elementary analogy to this phenomenon is a spring in parallel with a damper, with no threshold (Kelvin–Voigt model). Maxwell's series model, on the other hand, would rather describe a viscous fluid (Lemaître and Chaboche, 1985), since there is no recovery of the "inelastic" strain part (the damper).

Viscoelasticity is of little importance for metals and alloys. However, it is preponderant in polymers and all their compounds (especially organic matrix composites) and it also plays a role in concretes (in the long term).

For historical reasons, but also because they are simpler to begin with (due to the principles of linearity and superimposition), formulations of viscoelasticity are often based on integral expressions. We will recall the few basic elements and then go into the more general formulations (with a spectral type approach). Lastly, we will show the very close ties that can be established with a viscoplastic modeling used for metals.

4.1. *Integral formulations*

There is no question of giving a full and exhaustive presentation here. We will first assume the principle of linearity and that of Boltzmann superimposition in which the strain response to a sequence of increasing uniaxial stresses can be broken down into a sum such that:

$$\varepsilon(t) = \sum_j J(t-t_j)\,\Delta\sigma_j \tag{24}$$

in which $J(t)$ is the uniaxial creep function characteristic of the material's viscosity (i.e., the response to a unit step function stress input). Of course this equation is continuous and generalizes to:

$$\varepsilon(t) = \int_0^t J(t-\tau)\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}(\tau)\,\mathrm{d}\tau. \tag{25}$$

Clearly, the linearity means that for two proportional stress histories, $\sigma_2(t) = \lambda\sigma_1(t)$, the corresponding responses are in the same proportion at all times, with $\varepsilon_2(t) = \lambda\varepsilon_1(t)$. Let us note that relation (25) can be inverted to express the stress response from a given strain, by introducing a relaxation function. It is also obvious that these relations can also be generalized to a multiaxial situation for isotropic and anisotropic bodies (using the form of the instantaneous laws of elasticity).

The integral formulations are simple to write, to handle, and to identify (when they are appropriate). Yet their use is quite rapidly limited by the linearity hypothesis, which often enough does not hold, at least if we consider rather extensive variation domains. So many non-linear generalizations have been proposed, e.g., in the works of Souissi (1969), Sidoroff (1976), and Schapery (1969) for composite materials. However, for computing parts in practical applications, it is often necessary to integrate differential equations, which diminishes the advantage of the original integral formulation.

4.2. *Spectral formulations*

These consist in generalizing the sets of elementary rheological models directly by using a "spectrum" of variables. Each variable is associated with a specific "relaxation time". Of the many existing formulations (Nowick and Berry, 1972; Stango *et al.*, 1989), we will detail the one used recently by Maire (1992) for glass/epoxy composites.

Like for metals, the total strain breaks down into elastic and viscous components:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_v. \tag{26}$$

The spectral viscoelastic model consists in partitioning the viscous strain into a series of variables ζ_i:

$$\varepsilon_r = \sum_{i=n_1}^{n_2} \zeta_i \tag{27}$$

$$\dot{\zeta}_i = -\frac{1}{\tau_i}(\zeta_i - \mu_i \bar{\zeta}_i) \tag{28}$$

in which the variable $\bar{\zeta}$ will represent the asymptotic equilibrium states. The coefficients τ_i and μ_i are defined by the relaxation time spectrum. In the case of linear viscoelasticity, we can choose the asymptotic process as being linearly dependent on the stress:

$$\bar{\zeta}_i = a\sigma \quad \text{for any } i. \tag{29}$$

To reduce the number of free parameters in the model, we can use a continuous spectrum defined by a Gaussian distribution of the relaxation times, for example (Maire, 1992)

$$\tau_i = \exp(i) \qquad \mu_i = \frac{1}{\sqrt{\pi n_0}} \exp\left(-\left(\frac{i-n_c}{n_0}\right)^2\right). \tag{30}$$

The "viscous" part of the model is then defined by the parameters n_1, n_2, n_o, and n_c alone, and of course this behaves linearly. If the asymptotic process is chosen with a linear relation (29), the response of this model is then linear. For a constant stress $\sigma = \sigma_o$, we easily find by integrating:

$$\varepsilon_r = \sum_i a\mu_i \left[1 - \exp\left(-\frac{t}{\tau_i}\right)\right]\sigma_o. \tag{31}$$

One way of introducing the non-linearity consists in choosing a non-linear asymptotic behavior, i.e., non-linear relations between σ and $\bar{\zeta}_i$, such as:

$$\bar{\zeta}_i(\sigma) = \frac{1}{q_i} \|\sigma\|^{n-1}\sigma \tag{32}$$

in which n will characterize the non-linearity. Unfortunately, a non-linearity of this type is bi-unique and leads to abnormal load/unload properties such as in Fig. 1, for glass/epoxy [±45°] composite, taken from the thesis of Maire (1992). In order to recover more correct responses, one possibility, used by Maire, was to incorporate "updating rules" in the model. The curves simulated in Figs 3(b) and 5(b) are taken from this work. However, this approach is difficult to extend for general and complex cyclic loading conditions and it is

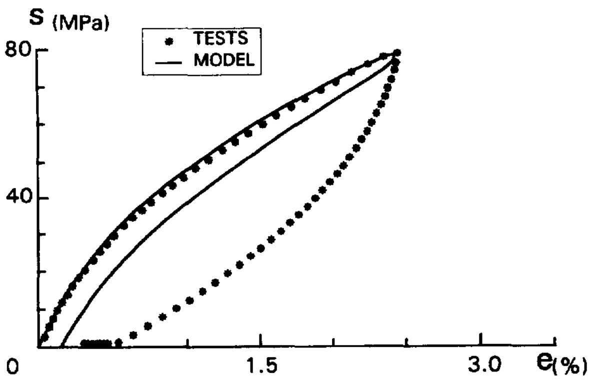

Fig. 1. Tensile test with unloading ($\dot{\sigma} = 0.5$ MPa/s) on a laminated glass–epoxy (±45°). **** Experiment, —— non-linear spectral model (Maire, 1992).

expected that more research is needed in order to develop acceptable cyclic viscoelastic spectral models.

4.3. *Link between viscoelasticity and viscoplasticity*

In theory, though the names for these properties differ in spelling by only one letter, they seem to be difficult to reconcile. From a physical point of view, this is obvious. In terms of macroscopic constitutive equations, it is less so. Firstly, we will examine two special cases of each of the theories and show that they are equivalent. Then we will see how this equivalence can be extended to more general situations. Let us note the similar discussions made in the past by Haupt (1992). However, one specificity of the present work is to show how a viscoplastic model classically used for metals is able to reproduce typical viscoelastic responses given by a spectral model.

Linear case. The elastoviscoplasticity scheme presented in Section 3.4, can be particularized as follows, by considering:

- a law without threshold and without isotropic hardening, $k = R = R^* = 0$, in eqn (2);
- linear viscosity, $n = 1$, still in the same equation;
- linear kinematic hardening, $D = 0$ in (21).

We limit ourselves here to the case of uniaxial tensile loading. The above hypotheses yield:

$$\dot{\varepsilon}_p = \frac{\sigma - c\varepsilon_p}{K} \tag{33}$$

for the plastic strain rate. The spectral formulation of viscoelasticity, described briefly in Section 4.2, can also be particularized by selecting a single variable ζ and, still in uniaxial loading mode, we obviously get:

$$\dot{\varepsilon}_v = \dot{\zeta} = \frac{1}{\tau}(\mu a \sigma - \varepsilon_v). \tag{34}$$

It is then easy to see the equivalence between the two theories, with $c = 1/\mu a$ and $K = \tau/\mu a$. Let us immediately note that this result easily generalizes to the multiaxial case. So in the case of linear viscosity and hardening, the two approaches are strictly equivalent.

Non-linear case. Here we use the viscoplastic formulation with a non-linear kinematic hardening law, but without threshold ($k = R = R^* = 0$) and with linear viscosity. The equations of Section 3.4, in uniaxial mode lead to:

$$\varepsilon = \varepsilon_e + \varepsilon_p \tag{35a}$$

$$\dot{\varepsilon}_p = \frac{\sigma - X}{K} \qquad X = \Sigma X_i \tag{35b}$$

$$\dot{X}_i = C_i\dot{\varepsilon}_p - D_i X_i \,|\, \dot{\varepsilon}_p \,| - \Gamma_i(X_i)X_i \tag{35c}$$

in which $\Gamma_i(X_i)$ is a function to be defined.

Properties simulated by this type of equation have a character very close to that of a viscoelastic material. Firstly, with $\Gamma_i(X_i) \equiv 0$, i.e., without static recovery of the hardening, we can simulate rather well the shape of the load–unload hysteresis cycles observed in

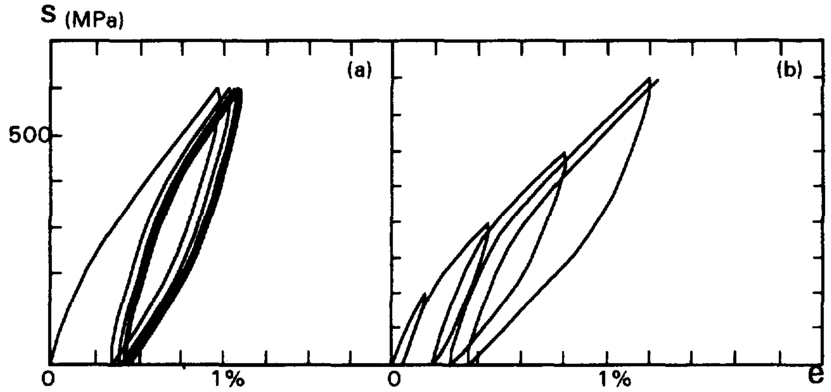

Fig. 2. Simulation of cyclic tensile loadings with nonlinear kinematic models adapted to the "viscoelastic" case. (a) Five periodic cycles, (b) cycles at increasing levels ($\dot{\sigma} = 600$ MPa/s).

polymer type materials (Fig. 2). Similarly, Fig. 3(a) shows curves with three set stress rates that can be reasonably compared, at least qualitatively, to the experiments of Fig. 3(b), taken from Maire's thesis (1992) for a glass/epoxy composite, as well as with simulations of a viscoelastic model based on the spectral formulation and updating rules.

We observe a major hysteresis, even when the stresses are always positive, which is reproduced thanks to the presence of a first variable that is nearly linear but with a very high modulus. The following coefficients were used for these simulations:

C_i:	60000	600000	25000	6000	1200	500	(MPa)
D_i:	1	5000	200	50	10	5	

We note that the comparisons made here are purely qualitative, as the stress and time scales are totally different.

In the case where constant-stress hold times are introduced (100 s here), creep occurs, but this stops when the variable X approaches the value σ (Fig. 4(a) in plastic stress–strain). So, after unloading and holding, the recovery of the strain is only partial, and consequently the material does not respond truly as viscoelastic. On the other hand, when a (time) static recovery term is present in eqn (35c), the behavior is viscoelastic. Figures 4(b) and 5(a) were simulated considering the following expressions for the functions Γ_i:

$$\Gamma_1(X_1) = r_1 D_1^m \left(\frac{|X_1|}{C_1}\right)^{m-1} \qquad (36a)$$

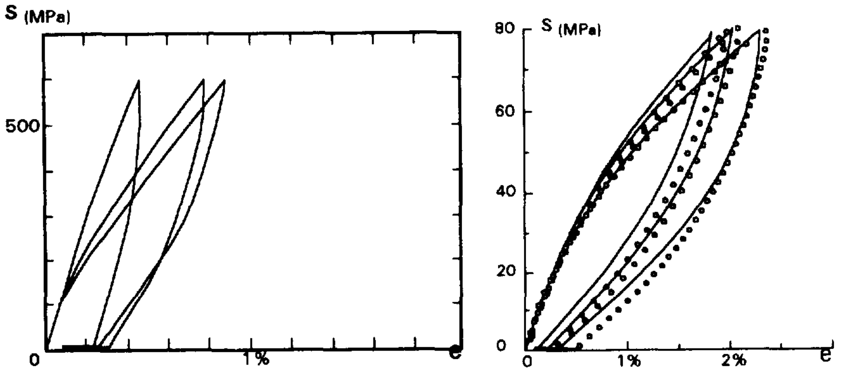

Fig. 3. Simulation of tensile load–unload tests at different rates. (a) Simulations with non-linear kinematic models and recovery ($\dot{\sigma} = 6000$, 600, 60 MPa/s), (b) experiments on glass–epoxy composite ($\dot{\sigma} = 50$, 5, and 0.5 MPa/s) and simulations with the spectral model (Maire, 1992).

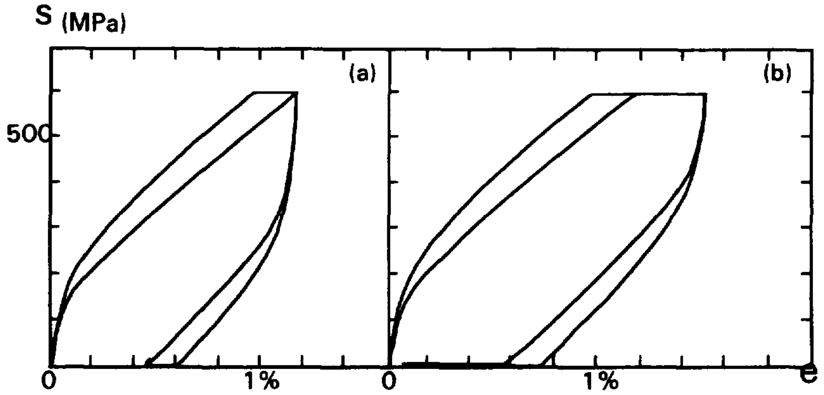

Fig. 4. Simulation of loadings with creep (100 s) and recovery (100 and then 1000 s). The inner curve represents the kinematic variable X. (a) With no static recovery of kinematic variable effect and (b) with static recovery effect.

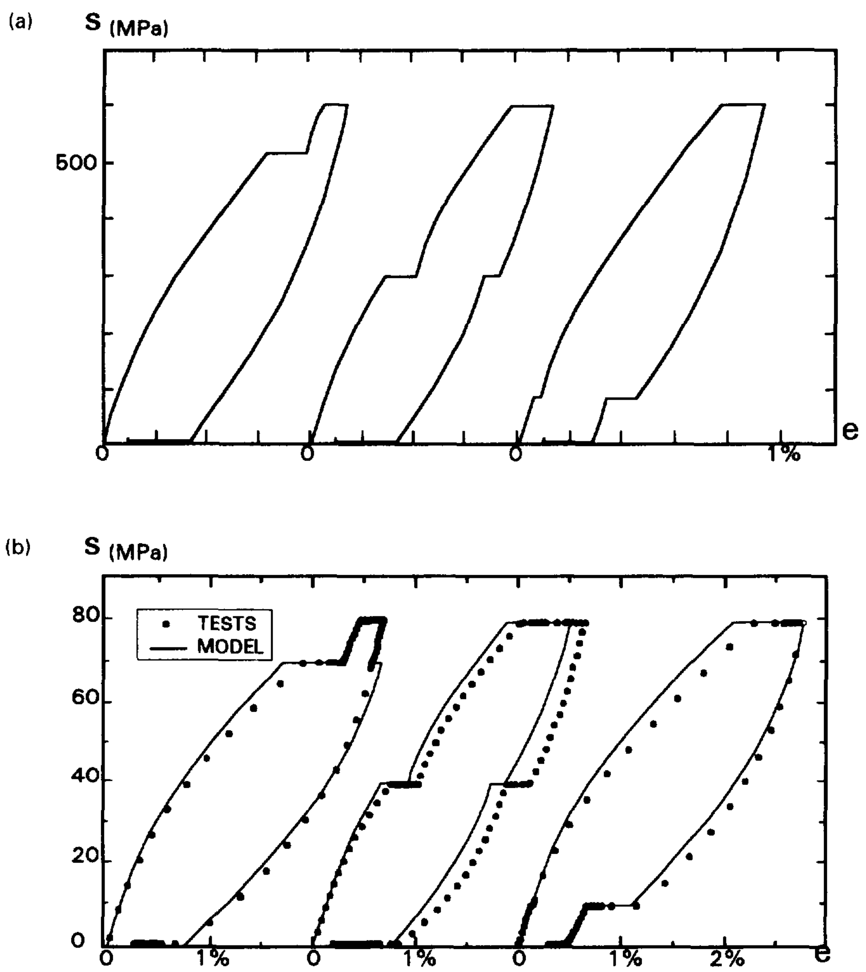

Fig. 5. (a) Simulations of tensile tests with hold times (10 s) at various levels during the loading, at the maximum and during unloading, followed by recovery (200 s), (b) tests on glass–epoxy composites (hold times of 1000 s) and simulation with the spectral model.

$$\Gamma_i(X_i) = r_2 D_i^m \left(\frac{|X_i|}{C_i}\right)^{m-1} \quad \text{for } i \geqslant 2. \tag{36b}$$

The coefficients chosen are the following: $m = 1.2$, $r_1 = 0$, $r_2 = 0.00002$. In these figures, we clearly see that the creep is more pronounced, thanks to the gradual recovery of

X, but also that the recovery of the initial strain is nearly complete after the unloading (after 1100 s). This clearly corresponds to a non-linear viscoelastic behavior. Moreover, if need be, the viscosity function of this viscoplastic type of model can easily be made non-linear again, with $n \neq 1$, which gives additional degrees of freedom with respect to the spectral formulation, which is always linear.

Lastly, Fig. 5(a) shows three simulations with hold times at the cycle extrema and in the course of transitory loading (at 1/8, 1/2, and 7/8 of the load). Depending on the case, the creep is more or less pronounced over the same periods of time. The results obtained are qualitatively compatible with experiments performed on a glass–epoxy composite (Fig. 5(b)).

5. CONCLUSION

Constitutive equations ordinarily used for describing cyclic viscoplasticity in metals were explained. These are compatible with a general thermodynamic formalism, which endows them with a theoretical foundation and possibilities for use in terms of stored and heat dissipated energies (Chaboche, 1993). The formalism presented here corresponds to a recent extension of the generalized standard materials concept (Chaboche *et al.*, 1995).

Moreover, these models now include the description of a set of phenomena occurring in metals, such as hardening, static recovery, ratchetting effects, which makes them highly useful for complex uniaxial or multiaxial cyclic loadings, with or without hold times (at high temperature). Let us note that these equations are specially intended for cyclic loadings with low strains.

Independently, the formulations usually used for describing the viscoelastic properties of polymers were reviewed very briefly. Usually, these models of linear or non-linear viscoelasticity are validated only for monotonic loadings (Maire, 1992).

Finally, we showed how the two types of formulations could be equivalent. The models of viscoplasticity (for metals) can be particularized into non-linear viscoelastic models (polymers) by eliminating the plasticity threshold, choosing a linear or slightly non-linear viscosity, and introducing a possibility of complete recovery of the hardening variables. The qualitative comparisons with experiments on glass–epoxy composites showed the possibilities of such modifications. Let us remark in particular their ability to describe the loading reversals, which are not correctly modeled by the classical spectral viscoelastic theories.

So, in theory, the phenomenological laws used for metals seem to be entirely transposable and usable for polymers. They still remain to be identified, of course, on precise cases, and the specific features of these materials have to be introduced (volumetric effects, different multiaxial effects, and so forth), in a proper large deformation description, along with the additional physical phenomena such as aging, hydrolysis, UV radiation effects, and so forth. Furthermore, the damage processes could be modeled using the formal framework of damage mechanics.

REFERENCES

Benallal, A. (1989). Thermoviscoplasticité et endommagement des structures. Thèse de Doctorat d'Etat, Univ. Paris VI.

Bodner, S. R. and Partom U. (1975). Constitutive equations for elastic viscoplastic strain-hardening materials. *Transactions of the ASME Journal of Applied Mechanics* **42**, 385–389.

Chaboche, J. L. (1983) Sur les lois de comportement des matériaux sous sollicitations monotones ou cycliques. *La Recherche Aérospatiale* **5**, 363–375.

Chaboche, J. L. (1989). Constitutive equations for cyclic plasticity and cyclic viscoplasticity. *International Journal of Plasticity* **5**, 247–302.

Chaboche, J. L. (1993). Cyclic viscoplastic constitutive equations, Part I: a thermodynamically consistent formulation; Part II: Stored energy—comparison between models and experiments. *ASME Journal of Applied Mechanics* **60**, 813–828.

Chaboche J. L., El Mayas, N. and Paulmier, P. (1995). Modélisation thermodynamique des phénomènes de viscoplasticité, restauration et vieillissement. *C.R. Academy of Science Paris*, **t.320**, Série II, 9–16.

Chow, C. L. and Wei, Y. (1991). A model of continuum damage mechanics for fatigue failure. *International Journal of Fracture* **50**, 301–316.

Germain P. (1973). *Cours de Mécanique des Milieux Continus*, Vol. 1, Masson, Paris.

Germain, P., Nguyen Q. S. and Suquet, P. (1983). Continuum thermodynamics. *ASME Journal of Applied Mechanics* **50,** 1010–1020.

Halphen, B. and Nguyen, Q. S. (1975). Sur les matériaux standard généralisé. *Journal de Mécanique* **14,** 39–63.

Hansen, N. R. and Schreyer, H. L. (1994). A thermodynamically consistent framework for theories of elastoplasticity coupled with damage. *International Journal of Solids and Structures* **31,** 359–389.

Hansen, N. R. and Schreyer, H. L. (1995). Damage deactivation. *ASME Journal of Applied Mechanics* **62,** 450–458.

Haupt, P. (1992). *Thermodynamics of Solids, in Non-equilibrium Thermodynamics with Application to Solids,* C.I.S.M. Courses, no. 59, Udine, Italy.

Ladevèze, P. (192). New advances in the large time increment method. In *New Advances in Computational Structural Mechanics* (eds P. Ladevèze and O. C. Zienkiewiecz), Elsevier, Oxford, pp. 3–22.

Lemaître J. and Chaboche, J. L. (1985). *Mécanique des matériaux solides.* In *Dunod.* (2nd ed. 1988), English version published by Cambridge University Press, Cambridge, 1990.

Maire, J. F. (1992). Etude théorique et expérimentale du comportement de matériaux composites en contraintes planes. Thèse de doctorat, Université de Besançon.

Mandel, J. (1971). *Plasticité Classique et Viscoplasticité,* C.I.S.M. Courses, Udine, Italy.

McDowell, D. L. (1992). A non-linear kinematic hardening theory for cyclic thermoplasticity and thermoviscoplasticity. *International Journal of Plasticity,* **8,** 695–728.

Miller, A. (1976). An inelastic constitutive model for monotonie cyclic and creep deformation. *Journal of Engineering Materials and Technology, Transactions ASME* **98,** 97–105 and 106–113.

Nouailhas, D. (1989). Unified modeling of cyclic viscoplasticity: application to austenitic stainless steel. *International Journal of Plasticity* **5,** 501–520.

Nouailhas, D. (1990). Un modèle de viscoplasticité cyclique pour matériaux anisotropes à symétrie cubique, *C.R. Academy of Science, Paris,* **t.310,** série II.

Nowick A. S. and Berry B. S. (1972). *Anelastic Relaxation in Crystalline Solids,* Academic Press, New York.

Ohno, N. (1990). Recent topics in constitutive modeling of cyclic plasticity and viscoplasticity. *Applied Mechanics Review* **43,** 283.

Schapery, R. A. (1969). On the characterization of non-linear viscoelastic materials. *Journal of Polymer Engineering Science* **9,** 295–310.

Sidoroff, F. (1976). Variables internes en viscoélasticité et viscoplasticité. Thèse de doctorat d'etat, Uni. de Paris VI.

Souissi, A. (1969). Description du comportement des métaux à chaud à l'aide d'une loi intégrale. Thèse, Uni. de Paris VI.

Stango, R. J., Wang, S. S. and Nelson, C. R. (1989). A note on analytical representation of anisotropic viscoelastic constitutive equations for fiber-reinforced composites, *Computer Science and Technology* **35,** 273–282.

Walker, K. P. (1981). Research and development program for non-linear structural modeling with advances time-temperature dependent constitutive relationships. *Report PWA*-5700-50, *NASA CR*-165533, NASA, U.S.A.

APPENDIX 1: CONCERNING THE USE OF MULTIPLE POTENTIALS FOR VISCOPLASTICITY WITH RECOVERY

To bring out the difference that exists between the broadened form, with multiple potentials, given in Section 3.3, and the usual "generalized standard materials" form, we can restrict ourselves to the case of viscoplasticity with or without static recovery. Below, we review the case with static recovery, to give a more general view; but the real difficulty (and the real difference), as we know, stems from the time-independent recall term in the non-linear kinematic hardening equation (or dynamic recovery term).

We use the following condensed notation: $\boldsymbol{\alpha}$ for hardening variables, $\mathbf{X} = \partial\psi/\partial\boldsymbol{\alpha}$ the associated thermodynamic forces, Ω^* the dissipation potential, expressed in the state variable rate space: $\Omega^*(\dot{\boldsymbol{\varepsilon}}_p, -\dot{\boldsymbol{\alpha}}; \boldsymbol{\alpha}, T)$. We note that the potential may depend on the state variables themselves (Germain *et al.*, 1983), and this is one of the keys to the conventional generalized standard materials form used below.

The inequality of the Second Principle reduces here to the positivity of the intrinsic dissipation (assumed to be decoupled from the thermal dissipation):

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p - \mathbf{X} : \dot{\boldsymbol{\alpha}} \geqslant 0. \tag{A-1}$$

The idea of generalized potential and generalized standard materials is associated with the rule of generalized normality, which is:

$$\boldsymbol{\sigma} = \frac{\partial\Omega^*}{\partial\dot{\boldsymbol{\varepsilon}}_p} \qquad \mathbf{X} = -\frac{\partial\Omega^*}{\partial\dot{\boldsymbol{\alpha}}}. \tag{A-2}$$

We convert to the dual potential in the thermodynamic forces space by the Legendre–Fenschel transform:

$$\Omega(\boldsymbol{\sigma}, \mathbf{X}) = \inf_{\dot{\boldsymbol{\varepsilon}}_p, -\dot{\boldsymbol{\alpha}}}\{\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p - \mathbf{X} : \dot{\boldsymbol{\alpha}} - \Omega^*(\dot{\boldsymbol{\varepsilon}}_p, -\dot{\boldsymbol{\alpha}})\} \tag{A-3}$$

and then easily show that the generalized normality is expressed:

$$\dot{\boldsymbol{\varepsilon}}_p = \frac{\partial \Omega}{\partial \boldsymbol{\sigma}} \qquad \dot{\boldsymbol{\alpha}} = \frac{\partial \Omega}{\partial \mathbf{X}}. \tag{A-4}$$

The hypotheses of potential convexity, positiveness, and the fact that they cancel out at the origin, easily show that the Second Principle (A-1) is automatically verified:

$$\mathscr{D} = \sigma : \frac{\partial \Omega}{\partial \boldsymbol{\sigma}} + \mathbf{X} : \frac{\partial \Omega}{\partial \mathbf{X}} \geqslant \Omega \geqslant 0. \tag{A-5}$$

Now here is the difficulty for the non-linear kinematic hardening laws. In the simple case of a viscoplasticity obeying a power function, the potential intuitively takes the form:

$$\Omega = \frac{K}{n+1} \left\langle \frac{f}{K} \right\rangle^{n+1} \tag{A-6}$$

in which $f = f(\boldsymbol{\sigma} - \mathbf{X})$ gives the elastic domain if $f < 0$ (e.g., (15) in Section 3.4), and defines the viscous stress if $f > 0$. Unfortunately, we then only find the linear kinematic hardening:

$$\dot{\boldsymbol{\varepsilon}}_p = \left\langle \frac{f}{K} \right\rangle^n \frac{\partial f}{\partial \boldsymbol{\sigma}} \qquad \dot{\boldsymbol{\alpha}} = -\left\langle \frac{f}{K} \right\rangle^n \frac{\partial f}{\partial \mathbf{X}} = \dot{\boldsymbol{\varepsilon}}_p. \tag{A-7}$$

To reach non-linear hardening in purely standard fashion, we have to add the quantity $\omega(\mathbf{X})$ to f (where ω is for example quadratic in $\mathbf{X}$, as in the expression (13)):

$$\Omega = \frac{K}{n+1} \left\langle \frac{f + \omega(\mathbf{X})}{K} \right\rangle^{n+1}. \tag{A-8}$$

It is this form that Ladevèze (1992) has chosen since a few years in using our cyclic viscoplasticity models. We do not wish to use it because it calls for an in-depth modification of the basic phenomenological behavior of the model, and demands a very high plastic flow threshold in the initial state with zero hardening (see Appendix 2).

The standard form usually adopted (Chaboche, 1983; Germain *et al.*, 1983; Lemaitre and Chaboche, 1985) then consists of subtracting the term $\bar{\omega}(\boldsymbol{\alpha})$, in which $\boldsymbol{\alpha}$ acts as a parameter, an important role pointed out above. We then have:

$$\Omega = \frac{K}{n+1} \left\langle \frac{f + \omega(\mathbf{X}) - \bar{\omega}(\boldsymbol{\alpha})}{K} \right\rangle^{n+1} + \Omega_s(\mathbf{X}) \tag{A-9}$$

in which we have also added a static recovery potential $\Omega_s(\mathbf{X})$. We now have, according to (A-4):

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}_p \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{A-10}$$

$$\dot{\boldsymbol{\alpha}} = -\dot{\lambda}_p \left(\frac{\partial f}{\partial \mathbf{X}} + \frac{\partial \omega}{\partial \mathbf{X}} \right) - \frac{\partial \Omega_s}{\partial \mathbf{X}} = \dot{\boldsymbol{\varepsilon}}_p - \frac{\partial \omega}{\partial \mathbf{X}} \dot{\lambda}_p - \frac{\partial \Omega_s}{\partial \mathbf{X}}, \tag{A-11}$$

a form that contains both the non-linear hardening or dynamic recovery (term in $\dot{\lambda}_p$) and the static recovery. In these expressions, we have let $\dot{\lambda}_p = \partial \Omega / \partial f$. The somewhat artificial key to the approach is that now we can say that:

$$\omega(\mathbf{X}) - \bar{\omega}(\boldsymbol{\alpha}) = 0 \tag{A-12}$$

in the state law, relating $\mathbf{X}$ and $\boldsymbol{\alpha}$, by adequately choosing the expression for the function $\bar{\omega}(\boldsymbol{\alpha})$, from the expression for $\omega(\mathbf{X})$. We then do find exactly the law desired, which is:

$$\dot{\lambda}_p = \frac{\partial \Omega}{\partial f} = \left\langle \frac{f}{K} \right\rangle^n. \tag{A-13}$$

This approach is said to be standard, and is considered to be part of the generalized standard materials framework (Germain *et al.*, 1983). For our part, we prefer the slightly broader framework defined in Section 3.3, which gets around the use of the device of (A-12). Let us recall the principle: we assume there to be several independently defined potential functions (we will no longer have $\Omega = \Omega_p + \Omega_s$), and several independent multipliers, $\dot{\lambda}_p$, $\dot{\lambda}_s$, known functions of variables such as:

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}_p \frac{\partial \Omega_p}{\partial \boldsymbol{\sigma}} + \dot{\lambda}_s \frac{\partial \Omega_s}{\partial \boldsymbol{\sigma}} \tag{A-14}$$

$$-\dot{\boldsymbol{\alpha}} = \dot{\lambda}_p \frac{\partial \Omega_p}{\partial \mathbf{X}} + \dot{\lambda}_s \frac{\partial \Omega_s}{\partial \mathbf{X}}. \tag{A-15}$$

Moreover, we assume that $\Omega_p = f(\boldsymbol{\sigma}-\mathbf{X})+\omega(\mathbf{X})$, in which f is still the yield surface, and $\Omega_s = \Omega_s(\mathbf{X})$ is independent of $\boldsymbol{\sigma}$. Further stating that $\dot{\lambda}_p = \langle f/K \rangle^n$ and $\dot{\lambda}_s = 1$, we obviously find:

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}_p \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{A-16}$$

$$\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\varepsilon}}_p - \frac{\partial \omega}{\partial \mathbf{X}} \dot{\lambda}_p - \frac{\partial \Omega_s}{\partial \mathbf{X}}. \tag{A-17}$$

It is clear that the generalized standard materials approach with a single potential (A-9) leads to expressions (A-16) and (A-17), and does it in a general writing. That is, we can always assume:

$$\Omega = \varphi_p(\Omega_p) + \varphi_s(\Omega_s) \tag{A-18}$$

and find (A-16) and (A-17) by letting $\dot{\lambda}_p = \varphi_p'$, $\dot{\lambda}_s = \varphi_s'$. On the other hand, the inverse is not true. We cannot work back from (A-16)–(A-17) to the generalized standard materials option, at least for the dynamic recovery term in the non-linear kinematic hardening law. The new approach proposed is slightly less restrictive. It still verifies the Second Principle automatically. However, it cannot be used to derive all the evolution equations from a single dissipation potential.

APPENDIX 2: WHY NOT USE THE LADEVÈZE GENERALIZATION OF THE NON-LINEAR KINEMATIC HARDENING?

As pointed out previously, Ladevèze (1992) proposes to use the dissipation potential (A-8) (in the visco-plasticity case without static recovery). He does not use the special device (A-12), in order to respect a purely standard generalized framework.

It is clear that the mechanical model is then modified, compared to (A-10), in that we now have a modified elastic domain defined by:

$$f^* = f + \omega(\mathbf{X}) = \|\boldsymbol{\sigma} - \mathbf{X}\| - k + \omega(\mathbf{X}) \leqslant 0 \tag{A-19}$$

if we neglect isotropic hardening ($R = 0$). The difficulty appears from the effective yield $k^* = k - \omega(\mathbf{X})$ in (A-19) that must be restricted to be always positive ($k^* \geqslant 0$) for obvious reasons. To respect this restriction we have only two ways:

- take a very large value of $k (k > \omega(X \max))$. This leads to a great modification of the mechanical response of the model compared to the reference version. Moreover, it increases greatly the initial value of the elastic limit (for $X = 0$), which is exactly the contrary that the line to follow if we want to respect experiments and to be able to describe a very smooth elastic–plastic transition at the beginning of the plastic flow.
- accept an experimental value for k, in that case k^* vanishes and must be maintained artificially at 0 for situations where $\omega(\mathbf{X}) > k$. In these situations, even if the mechanical response could be acceptable, the model is no longer working in the standard framework.

The difficulty is even increased when using the superposition of several back-stresses. This lack of flexibility is the main reason for us to not consider the Ladevèze generalization in our recent model developments.